

ON A REPRESENTATION OF MATHISSON-PAPAPETROU-DIXON EQUATIONS IN THE KERR METRIC

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Abstract

New representation of the exact Mathisson-Papapetrou-Dixon equations at the Mathisson-Pirani condition in the Kerr metric which does not contain the third-order derivatives of the coordinates of a spinning particle is obtained. For this purpose the integrals of energy and angular momentum of the spinning particle as well as a differential relationship following from the Mathisson-Papapetrou-Dixon equations are used. The form of these equations is adapted for their computer integration with the aim of further investigations of the influence of the spin-curvature interaction on the particle's behavior in the gravitational field without restrictions on its velocity and spin orientation.

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1. Introduction

In general relativity the two main approaches have been developed for the description of the spinning particle behavior in the gravitational field. Chronologically the first of them was initiated in 1929 when the usual Dirac equation was generalized for the curved space-time [1]. The second, the pure classical (non-quantum) approach, has been proposed in 1937 [2]. Later it was shown that in the certain sense the equations from [2] follow from the general relativistic Dirac equation as some classical approximation [3].

In the focus of this paper are the equations of motion of the classical spinning particle, which after [2] were obtained in [4] and later in many papers by different methods. These equations can be written as

$$\frac{D}{ds} \left(mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{ds} \right) = -\frac{1}{2} u^\pi S^{\rho\sigma} R^\lambda_{\pi\rho\sigma}, \quad (1)$$

$$\frac{DS^{\mu\nu}}{ds} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{ds} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{ds} = 0, \quad (2)$$

where $u^\lambda \equiv dx^\lambda/ds$ is the particle's 4-velocity, $S^{\mu\nu}$ is the tensor of spin, m and D/ds are, respectively, the mass and the covariant derivative with respect to the particle's proper time s ; $R^\lambda_{\pi\rho\sigma}$ is the Riemann curvature tensor (units $c = G = 1$ are used); here and in the following, Latin indices run 1, 2, 3 and Greek indices 1, 2, 3, 4; the signature of the metric $(-, -, -, +)$ is chosen.

Equations (1), (2) were generalized in [5] for the higher multipoles of the test particles and now set (1), (2) is known as the Mathisson-Papapetrou-Dixon (MPD) equations.

The first effects of the spin-gravity interaction following from (1)–(2) have been considered in [6] for the Schwarzschild field. According to [6] and by many further publications (some list of them is presented, for example, in [7, 8]) the influence of spin on the particle's trajectory is negligible small for practical registrations. However, in this sense much more realistic are the effects connected with the spin precession [9].

The interesting point has been elucidated in [10] concerning the possibility of the static position of a spinning particle outside the equatorial plane of the Kerr source of the gravitational field, on its axis of rotation. In spite of the conclusion that such situation is not allowed by the MPD equations [10], this question stimulated the investigations of possibilities of some non-static (dynamical) effects connected with the particle's motion relative to a Schwarzschild or Kerr mass outside the equatorial plane [11]. Then it was shown that spinning particles moving with relativistic velocity can deviate from geodesics significantly [12, 13].

While investigating the solutions of equations (1), (2), it is necessary to add a supplementary condition in order to choose an appropriate trajectory of the particle's center of mass. Most often the conditions [2, 14]

$$S^{\lambda\nu}u_\nu = 0 \quad (3)$$

or [5, 15]

$$S^{\lambda\nu}P_\nu = 0 \quad (4)$$

are used, where

$$P^\nu = mu^\nu + u_\lambda \frac{DS^{\nu\lambda}}{ds} \quad (5)$$

is the 4-momentum. The condition for a spinning test particle

$$\frac{|S_0|}{mr} \equiv \varepsilon \ll 1 \quad (6)$$

must be taken into account as well [10], where $|S_0| = \text{const}$ is the absolute value of spin, r is the characteristic length scale of the background space-time (in particular, for the Kerr metric r is the radial coordinate), and S_0 is determined by the relationship

$$S_0^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}. \quad (7)$$

In general, the solutions of equations (1), (2) under conditions (3) and (4) are different. However, in the post-Newtonian approximation these solutions coincide with high accuracy, just as in some other cases [16, 17]. Therefore, instead of exact MPD equations (1) their linear spin approximation

$$m \frac{D}{ds} u^\lambda = -\frac{1}{2} u^\pi S^{\rho\sigma} R^\lambda_{\pi\rho\sigma} \quad (8)$$

is often considered. In this approximation condition (4) coincides with (3) (by condition (3) in equations (1) $m = \text{const}$).

According to [18] for a massless spinning particle, which moves with the velocity of light, the appropriate condition is (3). Which condition is adequate for the motions of a spinning particle with the nonzero mass if its velocity is close to the velocity of light? To answer this question it is necessary to analyze the corresponding solutions of the exact MPD equations (1), (2) both at condition (3) and (4).

The main purpose of this paper is to consider the exact MPD equations under condition (3) in a Kerr metric. We use this metric in the Boyer-Lindquist coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$. There are Killing vectors $\vec{\xi}$ due to the symmetry of the Kerr space-time. As a result, equations (1), (2) have the constants of motion C_ξ :

$$C_\xi = \xi^\mu P_\mu - \frac{1}{2} \xi_{\mu;\nu} S^{\mu\nu},$$

and from it follows [19, 28–31]

$$E = P_4 - \frac{1}{2} g_{4\mu,\nu} S^{\mu\nu}, \quad (9)$$

$$J_z = -P_3 + \frac{1}{2} g_{3\mu,\nu} S^{\mu\nu}, \quad (10)$$

where E and J_z are the constants of the particle's energy and the projection of the angular momentum correspondingly.

If $S^{\mu\nu} = 0$, expressions (9), (10) pass to the known relationships for the geodesic motions which were effectively used for analyzing possible orbits of a spinless particle in a Kerr space-time [20, 21]. Namely, by the constants of energy and angular momentum the standard form of the geodesic equations, which are the differential equations of the second-order by the coordinates, can be reduced to the differential equations of the first order. Naturally, it is interesting to apply the similar procedure to the exact MPD equations using equations (9), (10). However, in contrast to the geodesic equations, the exact MPD equations at the condition (3) contain the third derivatives of the coordinates [22, 23]. Therefore, the application of this procedure to the exact MPD equations is not trivial.

In this paper for obtaining full set of the MPD equations at condition (3) without the third derivatives of the coordinates some differential relationship following from equations (1), (2) are used. We present this relationship in section 2 in general form, for any metric. Its concrete form in the Kerr metric, together with equations (9), (10), is used in section 3 and the full set of the differential equations for the dimensionless quantities connected with the particle's coordinates, velocity and spin is described. We analyze the relationship between u^λ and P^λ at condition (4) in section 4. We conclude in section 5.

It is known [22, 23] that in the Minkowski space-time the exact MPD equations under condition (3) have, in addition to usual solutions describing the straight worldlines, a set of solutions describing the oscillatory (helical) worldlines. The physical interpretation of these superfluous solutions was proposed by C. Möller [24]. He pointed out that in relativity the position of the center of mass of a rotating body depends on the frame of reference, and condition (3) is common for the so-called proper and non-proper centers of mass [24]. The usual solution describe the motion of the proper center of mass and the helical solutions describe the motions of the set of the non-proper centers of mass. Naturally, in general relativity, when the gravitational field is present, the exact MPD equations (1)–(3) have some superfluous solutions as well. Just to avoid these solutions, instead of (3) condition (4) was used in many papers. In contrast to (3), condition (4) picks out the unique worldline of a spinning particle in the gravitational field. However, the question arises: is this worldline close, in the certain sense, to the usual (non-helical) worldline of equations (1), (2) under condition (3)? It is simple to answer this question if the linear spin approximation is valid, because in this case condition (4) practically coincides with (3). Whereas another situation cannot be excluded

a priori for the high particle's velocity.

Note that the very condition (3) arose in a natural fashion in the course of its derivation by different methods [25–27]. Therefore, it is of importance to obtain a representation of the exact MPD equations at this condition in the Kerr metric convenient for their further computer integration.

We point out that the integrals of energy and angular momentum of the MPD equations in a Kerr space-time were effectively used for different purposes in [7, 28–36] at condition (4).

2. A relationship following from equations (1)–(3)

In addition to the antisymmetric tensor $S^{\mu\nu}$ in many papers the 4-vector of spin s_λ is used as well, where by definition

$$s_\lambda = \frac{1}{2} \sqrt{-g} \varepsilon_{\lambda\mu\nu\sigma} S^{\nu\sigma} \quad (11)$$

and g is the determinant of the metric tensor, $\varepsilon_{\lambda\mu\nu\sigma}$ is the Levi-Civita symbol. It follows from (7), (11) that $s_\lambda s^\lambda = S_0^2$ and at condition (3) we have $s_\lambda u^\lambda = 0$ (other useful relationships with s_λ following from MPD equations at different supplementary condition can be found, for example, in [8]).

The set of equations (2) contains three independent differential equations and in (3) we have three independent algebraic relationships between $S^{\lambda\nu}$ and u_μ . By (3) the components S^{i4} can be expressed through S^{ki} :

$$S^{i4} = \frac{u_k}{u_4} S^{ki}. \quad (12)$$

So, using (12) the components S^{i4} can be eliminated both from equations (2) and (1). That is, in further consideration one can "forget" about supplementary condition (3) and deal with the three independent components S^{ik} . However, it is appear that more convenient form of equations (1), (2) is not for S^{ik} but for another 3-component value S_i which is connected with S^{ik} by the simple relationship

$$S_i = \frac{1}{2u_4} \sqrt{-g} \varepsilon_{ikl} S^{kl}, \quad (13)$$

where ε_{ikl} is the spatial Levi-Civita symbol. For example, it is not difficult to check that three independent equations of set (2) in terms of S_i can be written as

$$u_4 \dot{S}_i + 2(\dot{u}_{[4} u_{i]} - u^\pi u_\rho \Gamma_{\pi[4}^\rho u_{i]}) S_k u^k + 2S_n \Gamma_{\pi[4}^n u_{i]} u^\pi = 0. \quad (14)$$

The simple calculation shows that the 3-component value S_i has the 3-vector properties relative to the coordinate transformations of the partial form $\hat{x}^i = \hat{x}^i(x^1, x^2, x^3)$, $\hat{x}^4 = x^4$ and in this special sense S_i can be called as a 3-vector. (By the way, in the context of equations (1), (2) firstly the 3-vector of spin was used in [6] with the notation $\mathbf{S} = (S^{23}, S^{31}, S^{12})$). By equations (11)–(13) the relationship between S_i and s_λ is

$$S_i = -s_i + \frac{u_i}{u_4} s_4. \quad (15)$$

Let us consider the first three equations of the subset (1) with the indexes $\lambda = 1, 2, 3$. Multiplying these equations by S_1, S_2, S_3 correspondingly and taking into account (11)–(13) we get

$$m S_i \frac{D u^i}{ds} = -\frac{1}{2} u^\pi S^{\rho\sigma} S_j R^j_{\pi\rho\sigma}. \quad (16)$$

We stress that in contrast to the each equation from set (1), which contain the third derivatives of the coordinates, relationship (16) does not have these derivatives.

Relationship (16) is an analog of relationship (21) from [8] where the spin 4-vector s_λ is used.

3. On set of exact MPD equations with constants of motion E, J_z for the Kerr metric

In the Boyer-Lindquist coordinates the non-zero components of the Kerr metric tensor are

$$\begin{aligned} g_{11} &= -\frac{\rho^2}{\Delta}, & g_{22} &= -\rho^2, \\ g_{33} &= -\left(r^2 + a^2 + \frac{2Mr a^2}{\rho^2} \sin^2 \theta\right) \sin^2 \theta, \\ g_{34} &= \frac{2Mr a}{\rho^2} \sin^2 \theta, & g_{44} &= 1 - \frac{2Mr}{\rho^2}, \end{aligned} \quad (17)$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \quad 0 \leq \theta \leq \pi.$$

It is convenient to use the dimensionless quantities y_i connected with the particle's coordinates, where by definition

$$y_1 = \frac{r}{M}, \quad y_2 = \theta, \quad y_3 = \varphi, \quad y_4 = \frac{t}{M}, \quad (18)$$

as well as the quantities connected with its 4-velocity

$$y_5 = u^1, \quad y_6 = Mu^2, \quad y_7 = Mu^3, \quad y_8 = u^4 \quad (19)$$

and the spin components [12]

$$y_9 = \frac{S_1}{mM}, \quad y_{10} = \frac{S_2}{mM^2}, \quad y_{11} = \frac{S_3}{mM^2}. \quad (20)$$

In addition, we introduce the dimensionless quantities connected with the particle's proper time s and the constants of motion E , J_z which is presented in (9), (10):

$$x = \frac{s}{M}, \quad \hat{E} = \frac{E}{m}, \quad \hat{J} = \frac{J_z}{mM}. \quad (21)$$

Quantities (18), (19) satisfy the four simple equations

$$\dot{y}_1 = y_5, \quad \dot{y}_2 = y_6, \quad \dot{y}_3 = y_7, \quad \dot{y}_4 = y_8, \quad (22)$$

here a dot denotes the usual derivative with respect to x .

Now we point out the seven nontrivial first-order differential equations for the 11 functions y_i . Namely, the first of them follows directly from equation (16). The second is a result of the covariant differentiation of the normalization condition $u_\nu u^\nu = 1$, that is

$$u_\nu \frac{Du^\nu}{ds} = 0. \quad (23)$$

The third and fourth equations follow from (9) and (10) correspondingly. Finally, the last three equations for y_i follow from (2) as written for the 3-vector of spin (we recall that for any metric the set of equations (2) contains three independent differential equations). Naturally, this set of the seven equations is too long and is not presented here for brevity. These seven equations together with the four equations from (22) are the full set of the exact MPD equations which describe most general motions of a spinning particle in the Kerr gravitational field without any restrictions on its velocity and spin orientation. We stress that the two equations following from (9) and (10) contain the quantities \hat{E} and \hat{J} as the parameters proportional to the particle's energy and angular momentum according to notation (21). By choosing different values of \hat{E} and \hat{J} for the fixed initial values of y_i one can describe the motions of different centers of mass. Among the set of the

pairs \hat{E} and \hat{J} there is the single pair corresponding to the proper center of mass. The possible approaches for finding this pair is a separate subject. One of them was proposed in [37] where a method of separation of some nonoscillatory solutions of the exact MPD equations in the Schwarzschild field was considered. In the next section we shall analyze the possibility of using condition (4) for the same purpose.

4. Values \hat{E} and \hat{J} according to condition (4)

Let us check the supposition that the single solutions of equations (1), (2) at the supplementary condition (4), corresponding to the fixed initial values of the coordinates, velocity and spin, is close to those solutions of equations (1), (2) at the condition (3) which describe the motion of the proper center of mass with the same initial values. As we pointed out in section 1, this assumption is justified for the velocities which are not close to the velocity of light. Here we shall consider the situation for any velocity.

Let us write the main relationships following from the MPD equations at condition (4) [28–31]. The mass of a spinning particle μ is defined as

$$\mu = \sqrt{P_\lambda P^\lambda} \quad (24)$$

and μ is the integral of motion, that is $d\mu/ds = 0$. The quantity V^λ is the normalized momentum, where by definition

$$V^\lambda = \frac{P^\lambda}{\mu}. \quad (25)$$

Sometimes V^λ is called the "dynamical 4-velocity", whereas the quantity u^λ from (1)–(3) is the "kinematical 4-velocity" [7]. As the normalized quantities u^λ and V^λ satisfy the relationships

$$u_\lambda u^\lambda = 1, \quad V_\lambda V^\lambda = 1. \quad (26)$$

There is the important relationship between u^λ and V^λ [28–30]:

$$u^\lambda = N \left[V^\lambda + \frac{1}{2\mu^2 \Delta} S^{\lambda\nu} V^\pi R_{\nu\pi\rho\sigma} S^{\rho\sigma} \right], \quad (27)$$

where

$$\Delta = 1 + \frac{1}{4\mu^2} R_{\lambda\pi\rho\sigma} S^{\lambda\pi} S^{\rho\sigma}, \quad (28)$$

Now our aim is to consider the explicit form of expression (27) for the concrete case of the Schwarzschild metric, for the particle motion in the plane $\theta = \pi/2$ when spin is orthogonal to this plane (we use the standard Schwarzschild coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$). Then we have

$$u^2 = 0, \quad u^1 \neq 0, \quad u^3 \neq 0, \quad u^4 \neq 0, \quad (29)$$

$$S^{12} = 0, \quad S^{23} = 0, \quad S^{13} \neq 0. \quad (30)$$

In addition to (30) by condition (4) we write

$$S^{14} = -\frac{P_3}{P_4}S^{13}, \quad S^{24} = 0, \quad S^{34} = \frac{P_1}{P_4}S^{13}. \quad (31)$$

Using (7), (29)–(31) and the corresponding expressions for the Riemann tensor in the Schwarzschild metric, from (27) we obtain

$$\begin{aligned} u^1 &= NV^1 \left(1 + \frac{3M}{r^3} V_3 V^3 \frac{S_0^2}{\mu^2 \Delta} \right), \quad u^2 = V^2 = 0, \\ u^3 &= NV^3 \left[1 + \frac{3M}{r^3} (V_3 V^3 - 1) \frac{S_0^2}{\mu^2 \Delta} \right], \\ u^4 &= NV^4 \left(1 + \frac{3M}{r^3} V_3 V^3 \frac{S_0^2}{\mu^2 \Delta} \right), \end{aligned} \quad (32)$$

where M is the mass of a Schwarzschild source. According to (28) we write the expression for Δ as

$$\Delta = 1 + \frac{S_0^2 M}{\mu^2 r^3} (1 - 3V_3 V^3) \quad (33)$$

(the quantity M in (32), (33) is the mass of a Schwarzschild source). Inserting (33) into (32) we get

$$\begin{aligned} u^1 &= \frac{NV^1}{\Delta} \left(1 + \frac{S_0^2 M}{\mu^2 r^3} \right), \\ u^3 &= \frac{NV^3}{\Delta} \left(1 - 2 \frac{S_0^2 M}{\mu^2 r^3} \right), \\ u^4 &= \frac{NV^4}{\Delta} \left(1 + \frac{S_0^2 M}{\mu^2 r^3} \right). \end{aligned} \quad (34)$$

In the following we use the notation

$$\varepsilon = \frac{|S_0|}{\mu r}, \quad (35)$$

where according to the condition for a test particle it is necessary $\varepsilon \ll 1$. However, in our calculations we shall keep all terms with ε .

The explicit expressions for N we obtain directly from the condition $u_\lambda u^\lambda = 1$ in the form

$$N = \Delta \left[\left(1 + \varepsilon^2 \frac{M}{r} \right)^2 - 3V_3 V^3 \varepsilon^2 \frac{M}{r} \times \right. \\ \left. \times \left(2 - \varepsilon^2 \frac{M}{r} \right) \right]^{-1/2}. \quad (36)$$

Inserting (36) into (34) we obtain the expression for the components V^λ through u^λ ($V^2 = u^2 = 0$):

$$V^1 = u^1 R \left(1 - 2\varepsilon^2 \frac{M}{r} \right), \\ V^3 = u^3 R \left(1 + \varepsilon^2 \frac{M}{r} \right), \\ V^4 = u^4 R \left(1 - 2\varepsilon^2 \frac{M}{r} \right), \quad (37)$$

where

$$R = \left[\left(1 - 2\varepsilon^2 \frac{M}{r} \right)^2 - 3(u^3)^2 \varepsilon^2 M r \left(2 - \varepsilon^2 \frac{M}{r} \right) \right]^{-1/2}. \quad (38)$$

The main feature of relationships (37), (38) is that for the high tangential velocity of a spinning particle the values V^1, V^3, V^4 become imaginary. Indeed, if

$$|u^3| > \frac{1}{\varepsilon \sqrt{6Mr}}, \quad (39)$$

in (38) we have the square root of the negative value. [As writing (39) we neglect the small terms of order ε^2 ; all equations in this section before (39) and after (40) are strict in ε .] Using the notation for the particle's tangential velocity $u_{tang} \equiv r u^3$ by (39) we write

$$|u_{tang}| > \frac{\sqrt{r}}{\varepsilon \sqrt{6M}}. \quad (40)$$

According to estimates similar to those which are presented in [13] if r is not much greater than M , the velocity value of the right-hand side of equation (40) corresponds to the particle's highly relativistic Lorentz γ -factor of order $1/\varepsilon$.

Probably, this fact that according to (25), (37)–(40) the expressions for the components of 4-momentum P^λ become imaginary is an evidence that condition (4) cannot be used for the particle's velocity which is very close to the velocity of light. However, this point needs some additional consideration. In any case, relationships (37)–(40) are of importance for authors which investigate solutions of the MPD equations at condition (4). We stress that many papers were devoted to study the planar or circular motions of spinning particles in the Schwarzschild or Kerr space-time at different supplementary conditions [6–13, 28–35, 38]. Equations (37)–(40) elucidate the new specific features which arise for the highly relativistic motions.

The question of momentum or velocity normalization for a spinning particle was discussed in [28, 29]. It is pointed out in [28] that there exist a critical distance of minimum approach of a spinning particle to the Kerr source where its velocity becomes space-like. The new result of this section as compare to [28, 29] consists in the conclusion that according to (37)–(40) only tangential component of velocity is important in this case, not the radial one, although the orbit is not necessarily circular.

It is interesting to check the possibility of using the values E and J_z as calculated by (37) for computing spinning particle motions by the equations which are described in the previous section if the particle's tangential velocity is much less than the critical value from the right-hand side of (40). At condition (4) the constants E and J_z for the equatorial motions in the Schwarzschild field can be written as

$$E = P_4 + \frac{1}{2}g_{44,1}S^{14} = \mu V_4 - \frac{1}{2}g_{44,1}\frac{V_3}{V_4}S^{13}, \quad (41)$$

$$J_z = -P_3 - \frac{1}{2}g_{33,1}S^{13} = -\mu V_3 - \frac{1}{2}g_{33,1}S^{13}. \quad (42)$$

Using the dimensionless quantities y_i as defined in (18), (19), relationship (37) and the simple expression for S^{13} through S_0 from (41), (42) we obtain

$$\hat{E} = R \left[y_8 \left(1 - \frac{2}{y_1} \right) \left(1 - \frac{2\varepsilon^2}{y_1} \right) + \varepsilon y_7 \left(1 + \frac{\varepsilon^2}{y_1} \right) \right], \quad (43)$$

$$\hat{J} = R \left[y_1^2 y_7 \left(1 + \frac{\varepsilon^2}{y_1} \right) + \varepsilon y_1 y_8 \left(1 - \frac{2}{y_1} \right) \left(1 - \frac{2\varepsilon^2}{y_1} \right) \right], \quad (44)$$

where similarly to (21) we note $\hat{E} = E/\mu$, $\hat{J} = J_z/\mu M$. Then for the fixed initial values of the quantities y_1, y_7, y_8 in (43), (44) we have the concrete values of \hat{E} and \hat{J} which can be used for numerical integration of the exact MPD equations. Some examples of such integration we shall consider in another paper.

5. Conclusions

In this paper we describe the representation of the exact MPD equations at supplementary condition (3) for a Kerr metric which can be obtained by using the constants of the particle's motion, the energy and angular momentum, together with the differential consequence of these equations (16). The way of obtaining the corresponding set of the 11 first-order differential equations is presented in section 3. The possibility using expressions (43), (44) in these equations to describe motions of a spinning particle in the Schwarzschild space-time we plan to consider in other publications, as well as to present results of complex investigating the highly relativistic motions of a spinning particle in the Kerr space-time according to the exact MPD equations.

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